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A DISC-SHAPED HYDRAULIC FRACTURE CRACK IN A POROELASTIC MEDIUM†

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The problem of a disc-shaped hydraulic fracture crack [1] in a fluid-saturated poroelastic medium is considered. The crack is opened by a flow of viscous fluid infiltrating into the fracture layer. The stressed state and strains of the poroelastic medium are described by the Biot equations [2]. The solution of the spatial axisymmetrical problem of a disc-shaped hydraulic fracture crack in a poroelastic medium is obtained by the methods of the theory of generalized analytic functions [3]. Unlike in an elastic medium, in a poroelastic medium crack opening is governed not only by the pressure of the fluid of the fracture at its sides, but also by the distribution of the rate of leakage of the fluid into the layer. An analytic solution is obtained for a stationary "ideal" crack along which the pressure is constant.

PLANAR problems of the theory of poroelastic media have been considered before in [4–6]. A representation of the general solution of the theoretical equations has been obtained in Papkovich–Neuber form [5], and those results were extended to weakly compressible liquids in [6]. A different approach, in which a representation of the general solution of plane problems is obtained in terms of two analytic functions, has been developed in [4]. Consideration has also been given [7–11] to the application of the theory of poroelastic media to the solution of problems of a hydraulic fracture of a fluid-saturated layer: stationary problems [7, 8], the planar non-stationary problem [9] and problems of Perkins–Kern cracks in poroelastic media [10, 11].

1. STATEMENT OF THE PROBLEM

Let an axisymmetric crack in an infinite fluid-saturated porous space and uniform compressive stress field σ_∞ be kept in the open state by the fluid discharged into it which, moving along the crack, can filter through its walls into the pore space. It is assumed that the radius of the hole r_0 is much smaller than the crack length ($L \ll r_0$), and so effects associated with the presence of the hole can be neglected.

This problem arises, in particular, in connection with the problem of a hydraulic fracture of an oil-bearing layer [1].

To describe the axisymmetric strain of a fluid-saturated porous medium and the filtration of a fluid in it, we will use the equations of coupled consolidation [2] in a cylindrical system of coordinates (i, j, $k = r$, Φ, z) with summation over repeated indices

$$\frac{\partial}{\partial r}\sigma_{rr} + \frac{\partial}{\partial z}\sigma_{rz} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} = 0, \qquad \frac{\partial}{\partial r}\sigma_{rz} + \frac{\partial}{\partial z}\sigma_{zz} + \frac{\sigma_{rz}}{r} = 0 \tag{1.1}$$

$$\sigma_{ij} = 2G\left(\varepsilon_{ij} + \frac{\nu}{1-\nu}\varepsilon_{kk}\delta_{ij}\right) - 2\eta\frac{1-\nu}{1-2\nu}p\delta_{ij} \tag{1.2}$$

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$$\frac{\partial}{\partial t}m+\frac{1}{r}\frac{\partial}{\partial r}(r\rho_0 u)+\frac{\partial}{\partial z}(\rho_0 v)=0,\quad \eta=\frac{3(\nu_u-\nu)}{2B(1-\nu)(1+\nu_u)} \tag{1.3}$$

$$m-m_0=\frac{\rho_0(1-\nu)}{G(1+\nu)}\eta\left(\sigma_{kk}+\frac{3}{B}p\right),\quad u=-\frac{k}{\mu}\frac{\partial}{\partial r}p,\quad v=-\frac{k}{\mu}\frac{\partial}{\partial z}p$$

$$\varepsilon_{rr}=\frac{\partial w_1}{\partial r},\quad \varepsilon_{\varphi\varphi}=\frac{w_1}{r},\quad \varepsilon_{zz}=\frac{\partial w_2}{\partial z},\quad \varepsilon_{r\varphi}=\varepsilon_{z\varphi}=0$$

$$\varepsilon_{rz}=\frac{1}{2}\left(\frac{\partial w_1}{\partial z}+\frac{\partial w_2}{\partial r}\right)$$

Here σ_{ij} is the total stress tensor, p is the pore pressure; B is Skempton's parameter, ν is Poisson's ratio, ν_u is Poisson's ratio corresponding to conditions where fluid cannot leave the medium [2], G is the shear modulus, ε_{ij} is the strain tensor, w_1 is the radial displacement, w_2 is the displacement in the direction of the axis of symmetry of the problem, m and m_0 are the mass of pore fluid per unit volume in the deformed and undeformed states, respectively, ρ_0 is the density of the fluid, k is the permeability coefficient, μ is the viscosity of the fluid and u and v are the components of the filtration velocity in the radial direction and in the direction of the axis of symmetry. The system of notation for the elastic constants proposed earlier in [4] is used.

Equations (1.1) and (1.2) are the equations of equilibrium and Hooke's law for a fluid-saturated porous medium. The relation between the flow of pore fluid and its mass is defined by the law of conservation of mass and Darcy's law.

The chosen cylindrical system of coordinates is such that the plane of the crack lies in the coordinate plane $z = 0$.

To describe the motion of discharged fluid along the hydraulic fracture crack, we use the equation of continuity and Poiseuille's law

$$\frac{\partial}{\partial t}w+\frac{1}{r}\frac{\partial}{\partial r}(rwu_c)=-2v_L,\quad u_c=-\frac{w^2}{12\mu}\frac{\partial}{\partial r}p_c \tag{1.4}$$

Here w is the opening of the crack sides, p_c is the pressure of the fluid discharged into the crack, u_c is the radial velocity of the fluid along the crack and v_L is the rate of fluid leakage through the crack wall into the porous medium.

The crack sides are subject to the boundary conditions

$$p_c(r,t)=p(r,z=0,t),\quad v_L(r,t)=\pm v(r,z=\pm 0,t) \tag{1.5}$$

2. METHOD OF SOLUTION

To solve the axisymmetric elastic problem, we will use the theory of generalized analytic functions [3]. We introduce complex variables and derivatives with respect to those variables

$$t=z+ir,\quad \bar{t}=z-ir,\quad \partial/\partial t={}^1/_2(\partial/\partial z-i\partial/\partial r),\quad \partial/\partial\bar{t}={}^1/_2(\partial/\partial z+i\partial/\partial r)$$

Then Eqs (1.1) and (1.2) can be transformed to the form

$$2\frac{\partial}{\partial t}F_*-\frac{F_*-\bar{F}_*}{t-\bar{t}}=2\frac{1-\nu}{1-2\nu}\frac{\partial}{\partial t}(\eta p) \tag{2.1}$$

$$F_*(t,\bar{t})=2G\left[\frac{1-\nu}{1-2\nu}\theta+i\omega\right],\quad \theta=\frac{\partial}{\partial r}u+\frac{u}{r}+\frac{\partial}{\partial z}w,$$

$$\omega=\frac{1}{2}\left(\frac{\partial}{\partial z}u-\frac{\partial}{\partial r}w\right)$$

Integrating (2.1), we obtain

$$F_*(t,\bar t) = \Phi_*(t,\bar t) + 2\frac{1-\nu}{1-2\nu}\eta p \tag{2.2}$$

where $\Phi_*(t,\bar t)$ is the generalized analytic function, that is, the function satisfying the equation $2\partial\Phi_*/\partial\bar t - (\Phi_* - \bar\Phi_*)/(\bar t - t) = 0$ [3].

From (2.1) and (2.2) we obtain an equation for the function $F(t,\bar t) = 2G(w - iu)$

$$2\frac{\partial}{\partial\bar t}F - \frac{F-\bar F}{t-\bar t} = \frac{\varkappa}{2(1-\nu)}\bar\Phi_* - \frac{1}{2(1-\nu)}\Phi_* + 2\eta p \tag{2.3}$$

Assuming that $p(r, z)$ is an analytic function with respect to the variables r, z, we integrate Eq. (2.3)

$$F(t,\bar t) = \varkappa\bar\Phi(t,\bar t) - 2z\frac{\partial}{\partial z}\Phi - \Psi + \eta\int d\bar t\, p(t,\bar t) \tag{2.4}$$

where Ψ is an arbitrary generalized analytic function and Φ is a generalized analytic function, related to Φ_* by the relation

$$\Phi_* = 2(1-\nu)\partial\Phi(t,\bar t)/\partial z = 2(1-\nu)\Phi'$$

Using expression (2.4) and Hooke's law, we obtain representations of the stress and strain fields of the saturated porous medium in terms of two generalized analytic functions

$$2G(\bar w - iu) - \eta\int d\bar t\, p(t,\bar t) = \varkappa\bar\Phi(t,\bar t) - 2z\Phi' - \Psi \tag{2.5}$$

$$\sigma_{rr} + \sigma_{zz} + \sigma_{\varphi\varphi} + 4\eta p = 4(1+\nu)\operatorname{Re}\Phi' \tag{2.6}$$

$$\sigma_{zz} - i\sigma_{rz} + \eta p - \eta\int d\bar t\,\partial p(t,\bar t)/\partial t = \Phi' - 2z\Phi'' - \Psi' \tag{2.7}$$

The derivative of the generalized function $\Phi' = \partial\Phi(t,\bar t)/\partial z$ can also be written in the form $\Phi' = -i\partial(\operatorname{Re}\Phi)/\partial r + r^{-1}\partial(r\operatorname{Im}\Phi)/\partial r$ [3] and so, after differentiating, (2.5) becomes

$$2G(r^{-1}\partial(ru)/\partial r + i\partial w/\partial r) - \eta p + \eta\int d\bar t\,\partial p(t,\bar t)/\partial t = \varkappa\bar\Phi' + 2z\Phi'' + \Psi' \tag{2.8}$$

To change from generalized analytic functions Φ, Ψ to ordinary analytic functions, we use the integral operators S and S^{-1} [3]

$$S\varphi = -\frac{1}{\pi|t-\bar t|}\int_{\bar t}^{t}\varphi(\sigma)M(\sigma,t)\,d\sigma;$$

$$S^{-1}\Phi = \frac{1}{2}\frac{d}{d\zeta}\int_{\bar\zeta}^{\zeta}h(\tau,\zeta)M(\zeta,\tau)\Phi(\tau)\,d\tau \tag{2.9}$$

$$(h(\tau,\zeta) = \operatorname{sign}(\operatorname{Im}\tau, \operatorname{Im}\zeta), \quad M(\sigma,t) = \sqrt{(\sigma-\bar t)/(\sigma-t)})$$

where $S^{-1}S\varphi = \varphi$ and $SS^{-1}\Phi = \Phi$; if Φ is a generalized analytic function, then $S^{-1}\Phi = \varphi$ is an ordinary analytic function.

After operating on expressions (2.7) and (2.8) with S^{-1}, after some algebraic manipulations we obtain

$$S^{-1}\left[\sigma_{zz} - i\sigma_{rz} + \eta p - \eta\int_{t_0}^{t}\frac{\partial p(t,\tau)}{\partial t}dt\right] = \varphi'(\xi) + 2(\xi - \bar\xi)\overline{\varphi''(\xi)} + \psi'(\bar\xi) \tag{2.10}$$

$$S^{-1}\left[2G\left(\frac{1}{r}\frac{\partial}{\partial r}(ru)+i\frac{\partial}{\partial r}w\right)-\eta p+\eta\int_{t_0}^{\tau}\frac{\partial p(t,\tau)}{\partial t}d\tau\right]=$$
$$=\varkappa\varphi'(\xi)-2(\xi-\bar{\xi})\overline{\varphi''(\xi)}-\psi'(\bar{\xi}) \tag{2.11}$$

where φ and ψ are ordinary analytic functions of the complex variable $\xi = x + iy (y = z)$.

We will formulate a mixed boundary-value problem for the analytic functions φ and ψ. We shall use the superposition principle, that is, we shall represent the stress and strain fields in the form of the sum of two fields, one of which corresponds to a continuous solid under the effect of loads applied within it (σ_∞ is a uniform compressive stress and p_∞ is the unperturbed pressure of the pore fluid) and the other, to a solid with a cut, to the surfaces of which loads are applied.

The rate of fluid leakage from the crack into the layer is equal to

$$v_{L}^{\pm}=\pm v=\mp\frac{k}{\mu}\frac{\partial}{\partial z}p(z=0\pm 0,r) \tag{2.12}$$

The operator S^{-1} associates with the spatial axisymmetric deformed state (z, r) a plane state which is symmetrical about $x = 0$ axis $(y = z, r \to x)$. Bearing in mind that the values of the radicals occurring in the operator S^{-1} on different sides of the cut differ in sign, and putting $t_0 = 0$, in order to satisfy the symmetry conditions (relative to the $x = 0$ axis), from (2.10) we obtain the boundary values of the analytic functions φ', ψ' $(y = z = 0 \pm 0)$

$$\pm R-iT=\varphi'^{\pm}(x)+\psi'^{\pm}(x)$$
$$R=S_0^{-1}\Sigma,\quad T=\left(\frac{\eta\mu}{2k}\right)S_1^{-1}\int_0^r v\,ds,\quad S_k^{-1}Q=\operatorname{sign}(x)\frac{d}{dx}\int_0^x\frac{x^k r^{1-k}Q(r)\,dr}{\sqrt{x^2-r^2}} \tag{2.13}$$
$$\Sigma(r)=p(r)-\sigma_\infty-\eta(p(r)-p_\infty)-{}^1\!/_2\eta(p(r)-p(0))$$

The solution of boundary-value problem (2.13) is known [12]. Taking into account the zero conditions at infinity and the conditions of symmetry about the $x = 0$ and $y = 0$ axes, this solution can be written in the form

$$\varphi'(\xi)-\psi'(\xi)=\frac{1}{i\pi}\int_{-l}^{l}\frac{R(x)\,dx}{x-\xi}+\frac{1}{i}\frac{A\xi}{(l^2-\xi^2)}$$
$$\varphi'(\xi)+\psi'(\xi)=\frac{1}{\pi\sqrt{\xi^2-l^2}}\int_{-l}^{l}\frac{\sqrt{x^2-l^2}\,T(x)\,dx}{x-\xi}\qquad(\xi=x+iy) \tag{2.14}$$

The presence of the second term in the first expression of (2.14) is due to the fact that the function R is undefined at the vertices of the section $x = \pm l$ (which were excluded from consideration when formulas (2.13) were obtained) and so, for the desired function $\varphi' - \psi'$, the presence of poles at the indicated points, the order of which depends on the order of the singularities in the neighbourhood of those points, must be allowed. In this case, the order of the poles is dictated by the root singularity of the stress tensor as $x \to \pm l$.

Allowing for the properties of the operator S, from (2.11) and (2.14) with $y = z = 0$ we obtain

$$\frac{\partial}{\partial r}w=\frac{\varkappa+1}{4G}\left\{-\int_0^r v\,ds+\frac{2}{\pi r}\int_0^r\frac{y\,dy}{\sqrt{r^2-y^2}}\times\right.$$
$$\left.\times\left[\frac{1}{\pi}\int_{-l}^{l}\frac{dx}{x-y}\frac{d}{dx}\int_0^x\frac{s\,ds}{\sqrt{x^2-s^2}}\Sigma(s)+\frac{Ay}{l^2-y^2}\right]\right\} \tag{2.15}$$

Integrating (2.15) with respect to r taking into account the boundary conditions on the crack sides and singularities near the crack tips, after simplifying we obtain

$$w = \frac{\varkappa + 1}{4G}\left\{\frac{2}{\pi}\int_r^l\int_0^x \frac{\Sigma(s)\, s\, dx\, ds}{\sqrt{x^2 - r^2}\sqrt{x^2 - s^2}} + W_v\right\} \tag{2.16}$$

$$W_v = -\frac{\eta\mu}{2k}\int_r^l d\rho \int_0^\rho v(s)\, ds = \frac{\eta\mu}{2k}\left[(l - r)\int_0^l v\, ds + \int_r^l v(r - s)\, ds\right]$$

$$A = -\frac{2}{\pi}\int_0^l s\Sigma(s)(l^2 - s^2)^{-1/2}\, ds \tag{2.17}$$

If the criteria of fracture of [13] are used, then the constant A can also be expressed in terms of the rock adhesion modulus K_I:

$$A = -\pi^{-1} K_I \sqrt{l/2}$$

From (2.6) and (1.3) we obtain an equation for the transport of pore fluid in the layer:

$$\frac{\partial}{\partial t} p = c\Delta p - \omega \frac{\partial}{\partial t}(\operatorname{Re}\Phi')\left(\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}\right) \tag{2.18}$$

$$c = \frac{1 + \nu}{1 - \nu}\frac{kGB}{\mu\eta(3 - 4\eta B)}, \quad \omega = \frac{4(1 + \nu)B}{3 - 4\eta B}$$

$$\operatorname{Re}\Phi' = \operatorname{Re} S\overline{\varphi'(\xi)} = -\frac{1}{\pi^2}\int_0^l \tau K(y, r; \tau)\Sigma(\tau)\, d\tau + \frac{\eta\mu}{2\pi^2 k} \times$$

$$\times \int_0^l L(y, r; \rho) v(\rho) \rho\, d\rho$$

$$K(y, r; \tau) = \int_\tau^l \int_0^r \frac{d\zeta\, ds}{\sqrt{\zeta^2 - r^2}\sqrt{r^2 - s^2}}\frac{d}{d\zeta}\operatorname{Im}\left[\frac{-2\zeta}{\zeta^2 - \xi^2}\right]$$

$$L(y, r; \rho) = -\rho\int_\rho^l \int_0^r \frac{\sqrt{l^2 - \zeta^2}\, d\zeta\, ds}{\sqrt{\zeta^2 - \rho^2}\sqrt{r^2 - s^2}}\operatorname{Re}\left[\frac{-2}{\sqrt{l^2 - \xi^2}(\xi^2 - \zeta^2)}\right]$$

$$(\xi = s + iy)$$

3. THE STEADY PROBLEM

Let us consider the steady problem of the hydraulic fracture of a porous fluid-saturated medium of an axisymmetric immobile crack $l = \text{const}$. In this case, the piezoconduction equation (2.18) reduces to Laplace's equation for the pore fluid pressure

$$\Delta p = 0 \tag{3.1}$$

Apart from the pressure distribution on its sides $p_0(r)$, the expression for the opening of a hydraulic fracture crack contains the rate of fluid leakage into the layer $v(r)$. On the crack sides, $v(r)$ can be found from solution of Laplace's equation (3.1) with given pressure $p_0(r)$.

Because the problem is symmetrical about the $y = 0$ plane, we shall formulate the boundary value problem for Laplace's equation (3.1) in the upper half-plane $y > 0$:

$$p(r, z=0) - p_\infty = p_0(r) - p_\infty, \ r < l; \ \partial p(r, z=0)/\partial r = 0, \ r > l$$

$$p(r, z \to \infty) - p_\infty = 0 \tag{3.2}$$

The boundary value problem (3.1), (3.2) in a cylindrical system of coordinates (r, z) can be solved by the method of dual integral equations [14]

$$p(r,z)=\frac{2}{\pi}\int_0^\infty d\lambda e^{-\lambda z}J_0(\lambda r)\int_0^l dt\cos\lambda t\frac{d}{dt}\int_0^t\frac{dss(p_0(s)-p_\infty)}{\sqrt{t^2-s^2}} \tag{3.3}$$

Since $v(r)=-(k/\mu)\partial p(r<1, z=0)/\partial z$, from (2.16) and (3.3) the contribution to crack opening from the leakage of fluid into the layer is

$$\begin{aligned}Wv=-\frac{\eta\mu}{2k}\int_r^l d\rho\int_0^\rho v(s)\,ds=\frac{\eta}{\pi}\Big[\frac{1}{l}\sqrt{l^2-r^2}\int_0^l\frac{s(p_0(s)-p_\infty)\,ds}{\sqrt{l^2-s^2}}-\\ -\int_0^r dss(p_0(s)-p_\infty)\int_r^l\frac{d\rho}{\sqrt{\rho^2-s^2}}G(\rho,r)-\\ -\int_r^l dss(p_0(s)-p_\infty)\int_s^l\frac{d\rho}{\sqrt{\rho^2-s^2}}G(\rho,r)\Big]\Big\}\end{aligned} \tag{3.4}$$

$$G(\rho,r)=\frac{\partial}{\partial\rho}\left(\frac{\sqrt{\rho^2-r^2}}{\rho}\right)$$

In the general case, the pressure distribution of the fluid in a crack can be found from the solution of the system of equations (1.4).

Consider the special case of a hydraulic fracture crack with high hydraulic conductivity. The pressure distribution in such a crack can be taken to be approximately constant along its sides.

$$p_0(r)=p_0=\text{const} \tag{3.5}$$

Substituting (3.5) into (3.4), we obtain

$$W_v=-\frac{\eta(p_0-p_\infty)l}{\pi}\left[\frac{r}{l}\arccos\left(\frac{r}{l}\right)-\sqrt{1-\left(\frac{r}{l}\right)^2}\right] \tag{3.6}$$

The total opening of an hydraulic fracture crack can be found from (2.16), using relations (3.5) and (3.6). We obtain

$$w(r)=\frac{\varkappa+1}{2\pi G}l\left\{(p_0-\sigma_\infty)\sqrt{1-\left(\frac{r}{l}\right)^2}-\eta(p_0-p_\infty)\sqrt{1-\left(\frac{r}{l}\right)^2}-\frac{W_v}{l}\right\} \tag{3.7}$$

The first term in (3.7) corresponds to the purely elastic solution. The second term is associated with the fact that, in a poroelastic medium, the load is transmitted to both the fluid and solid phases, with different moduli of compressibility. If $p_0>p_\infty$, this term produces a smaller crack opening than the elastic solution. And, finally, the third term is due to swelling of the poroelastic medium during filtration of the fluid from the crack into the medium $(v>0)$.

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INTEGRAL EQUATIONS FOR A THIN INCLUSION IN A HOMOGENEOUS ELASTIC MEDIUM†

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A study is made of equilibrium in a homogeneous elastic medium containing a thin inclusion whose elastic moduli differ substantially from those of the medium. The solution depends on two non-dimensional parameters: the ratio δ_1 of the characteristic linear dimensions of the inclusion and the ratio δ_2 of the elastic moduli of the inclusion and the medium. While δ_1 is always small, δ_2 may be either small or large. The problem of constructing the principal asymptotic terms of the elastic fields in the neighbourhood of a thin inhomogeneity based on these parameters has been reduced [1] to the solution of integral (pseudodifferential) equations on the middle surface of the inclusion. Similar equations are obtained with two-dimensional models of thin inclusions [2–5]. Some properties of the solutions of these equations will be discussed below. A method is proposed for the numerical solution of the equations, based on introducing a special class of approximating functions, thanks to which the problem can be reduced to a system of linear algebraic equations whose matrix can be calculated by analytical means. The idea of the method is due to V. G. Maz'ya.

1. INTEGRAL EQUATIONS FOR THIN DEFORMABLE AND RIGID INCLUSIONS

A HOMOGENEOUS elastic medium with tensor of moduli C_0 contains an inclusion that occupies a bounded region V, one of whose characteristic dimensions h is small compared with the other two (of order l), so that $\delta_1 = h/l$ is a small parameter. The inclusion is ideally connected to the medium

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